Report of the Minor Research Project

Title- **"Study of Self-Similar Gravitational Collapse of a superdense Star"**

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1 Gravitational Collapse of massive Star

If a star has mass higher than or about $5M_{\odot}$, it must enter a state of perpetual gravitational collapse once it has exhausted all its nuclear fuel and no equilibrium configurations are possible unless it manages to throw away most of its mass by some process during this evolution. In fact, mass ejection is observed in a *supernova* explosion for the star. When the core collapse is halted or slowed down at nuclear densities, a shock wave is produced which propagates outwards in the envelope of the star. While the inner core remains a neutron star, the outer parts are driven away by the shock releasing enormous mass and energy, which is believed to be a supernova explosion. The theory for such ejection of matter is not well understood, however, and at any rate it seems unlikely that all such massive stars will be able to throw away almost all of their mass in such a process. The reason is, for stars having tens of solar masses, this would amount to throwing away almost ninety percent of the mass of the star. No suitable mechanism is envisaged today which could achieve such a high degree of efficiency. Thus, if the shock could not blow off all the outer layers, they would fall on the newborn neutron star and the collapse would continue again.

It was thought that a massive star which has exhausted its fuel, would undergo mass ejection process and the mass of remnant star would always be lower than the limit mentioned for the star to become a *white dwarf* through electron degeneracy pressure, or from neutron degeneracy pressure giving a *neutron star* [1]. Chandrashekhar [2] approximated the equation of state in this case by an ideal electron Fermi gas and showed that there is a maximum mass limit for the mass of spherical non-rotating star to achieve a *white dwarf* stable state, which is given by $M_c \sim 1.4 \left(\frac{2}{\mu_c}\right)^2 M_\odot$ (where μ_c is the constant mean molecular weight per electron). Subsequently, considerable work has been done on equations of state for the matter at nuclear densities and it is seen that the maximum mass for non-rotating white dwarfs lies in the range $1.0M_\odot - 1.5M_\odot$ depending on the composition of matter. Similar considerations for neutron stars give corresponding range to be $1.3M_\odot - 2.7M_\odot$.

But for massive enough stars (and very massive objects like the center of galaxy) there is no evidence, either from the studies of stellar evolution or from observations, that the mass loss could possibly play any such role.

1.1 Spherically Symmetric Space-time

In a spherically symmetric space-time, if P is any point at a distance r from the origin O, the system must be invariant under rotations around O. Such rotations will generate a two-sphere around O and the line element on it is given by

$$ds^2 = r^2(d\theta^2 + sin^2\theta d\phi^2) \ . \tag{1}$$

This is a line element for a two-sphere given by $t = const., r = const.$ in a general spherically symmetric space-time. Further, as the metric must be invariant under the reflections $\theta \to \pi - \theta$ and $\phi \to -\phi$, there must not be any cross term in the metric in $d\theta$ and $d\phi$. As the line element must not change with any change in θ and ϕ, they must occur in the metric only in the form of the two-metric given above. Then, in the (t, r, θ, ϕ) coordinates systems, the metric has the form

$$ds^2 = -e^{\mu}dt^2 + e^{\nu}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{2}$$

where $\mu = \mu(t,r)$ and $\nu = \nu(t,r)$.

The matter fields of a spacetime are represented by the energy-momentum tensor (T_{ij}) which are connected with the spacetime metric through the Einstein's equations namely

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R = 8\pi T_{ij} \tag{3}$$

where G_{ij}, R_{ij}, R and T_{ij} are the Einstein tensor, the Ricci tensor, the Ricci scalar and the stress-energy tensor respectively. The all important conservation of energy and momentum are implied by the Bianchi identities namely

$$G^{ij}_{;j} = 0 \quad . \tag{4}$$

The Einstein's theory provides a vital interplay between the matter distribution in the spacetime and its geometry. The force of gravity is built into the theory from the outset through the Einstein's equations (3), which link the spacetime curvature to the distribution of matter.

The Einstein equations form a highly non-linear system of partial differential equations and due to their complexity, a complete general solution is not known. The known exact solutions usually assume a rather high degree of symmetry such as spherical or axial symmetry, or equivalently the existence of necessary Killing vector fields on the spacetime, and to that extent represent an idealized situation. However, such spacetime examples provide a good idea of what is possible within the framework of the general theory of relativity. For example, the Minkowski spacetime is both the geometry of the special theory of relativity and locally that of any general relativistic model. Also, the spherically symmetric and asymptotically flat spacetimes such as the Schwarzschild, radiating Schwarzschild (Vaidya radiation model) and Kerr geometries are useful to model the spacetime outside the Sun and stars and could be used to obtain conclusions relevant for the experimental tests of the general theory of relativity.

Such solutions could also possibly represent the outcome of a complete gravitational collapse of massive star.

1.2 Space-time singularity

By a spacetime singularity, we mean a portion of spacetime where mathematical equations of spacetime won't fit at all, and none of the known physical laws would be applicable to that portion of spacetime. End state of a collapsing star is thus an example of a singularity wherein the scalar curvature, and the physical parameters like density and pressure assume infinite values.

For general spherically symmetric collapse model, we usually have an initial Cauchy hypersurface S on which a non-singular initial data is specified. The spacetime outside S is taken to be Ricci flat and inside this region, we have the collapsing matter satisfying some physically reasonable equation of state. The Einstein equations are evolved in the future of S. For a certain set of initial conditions that include the formation of a trapped surface during the collapse, a spacetime singularity develops (that is maximal evolution of the field equations provides a spacetime which is geodesically incomplete). Since the curvature scalars such as $R_{ijkl}R^{ijkl}$ diverge to infinity at a finite affine parameter value, the model cannot be extended to a 'more' complete spacetime. In the process of such a collapse, a region H develops in the spacetime from which it is not possible to draw timelike curves of infinite length into the future so as to reach a far away observer. Inside H, any nonspacelike curve when extended maximally in future would encounter a spacetime singularity. The boundary of this region is called absolute *event horizon* and the region H is called a *black hole*(BH) in the spacetime. For further details refer [1].

1.3 Energy conditions

1. The weak energy condition (WEC)

The term $T_{ij}V^iV^j$ represents the energy density as measured by a timelike observer with the unit tangent vector V^i where T_{ij} and V^i are the energy momentum tensor and four velocity of the observer respectively. For all reasonable classical physical fields this energy density (ρ) is generally taken as non-negative. Therefore it is assumed that for all timelike vectors V^i, $T_{ij}V^iV^j \geq 0$ is satisfied. Such an assumption is called the *weak energy condition*. This leads to the condition that $\rho \geq 0$ and $\rho + p \geq 0$ where p is the pressure of the medium under consideration.

2. The strong energy condition (SEC)

It is reasonable that the matter stresses should not be so large that right hand side of equation(3) becomes negative and this is satisfied when $T_{ij}V^iV^j \geq -1/2T$. Such an assumption is called the *strong energy condition* and it implies that for all timelike and null vectors V^i, $R_{ij}V^iV^j \geq 0$. In this situation, we have $\rho \geq 0$ and $\rho + 3p \geq 0$. Both the WEC and SEC are valid for well known forms of matter such as perfect fluid.

3. The dominant energy condition (DEC)

This energy condition is required to study the *singularity theorems* which states that in addition to the WEC, the pressure of the medium must not exceed the energy density. In other words, for all timelike vectors V^i, $T_{ij}V^iV^j \geq 0$ and the vector $T^{ij}V_i$ is a non-spacelike vector. Such condition would be satisfied provided the local speed of sound does not exceed the local speed of light which means $dp/d\rho < c^2$ [3].

1.4 Cosmic Censorship Conjecture (CCC)

The singularity theorems do not necessarily imply that all the singularities developing in gravitational collapse must be covered by event/apparent horizon. The apparent horizon within the collapsing cloud is given by $F/R^{N-1} = 1$ (N is the dimension of the spacetime), which gives the boundary of the trapped surface region in the spacetime. It is the behaviour of the apparent horizon curve (which meets the central singularity at $R = r = 0$) near the centre which essentially determines the visibility, or otherwise, of the central singularity. For example, it is known within the context of the Tolman-Bondi-Lemaitre models that the apparent horizon can be either past pointing timelike or null, or it can be spacelike, as can be seen by examining the nature of the induced metric on this surface [4]. This is unlike the event horizon curve which is always future pointing and null. If the neighbourhood of the centre gets trapped earlier than the singularity, then it is covered (BH), and if that is not the case, then the singularity can be naked, with families of non-spacelike trajectories escaping from it. We may thus have initial data sets that can evolve into the formation of what is called a *naked singularity* (NS).

Naked singularities can communicate with faraway observers. We can have timelike and null trajectories coming out from such a singularity, and possibly escaping to future null infinity and thus reaching a faraway observer. Thus, obscure information can be gathered from naked singularities and so the usual way of making predictions in the gravitation theory may break down. Existence of naked singularities may also affect many of the standard assumptions and results in the black hole physics. To ensure avoidance of such an *unphysical* eventuality, one generally has to assume that the evolution of continual gravitational collapse from generic initial data is such that, the resulting spacetime singularity is always covered so that the standard predictability is preserved. This assumption is known as the *cosmic censorship conjecture* (CCC), articulated by Roger Penrose [5].

The, cosmic censorship conjecture can be stated in other words as
The result of a complete gravitational collapse must always be a black hole and not a naked singularity, or all singularities of collapse must be hidden inside the event horizon, causally disconnected from observers at infinity.

The CCC remains one of the most important unsolved problems in general relativity and gravitation theory today. According to the *s*trong cosmic censorship conjecture, the singularities that appear in gravitational collapse are hidden from all observers. It is weakened to exclude only *locally* naked singularities in the sense that an observer within the event horizon and in the interior of the black hole could possibly receive particles or photons from the singularity, but may well be hidden behind an event horizon as opposed to *globally* naked singularities visible to an asymptotic observer. So, if the outgoing non-spacelike curves emanated from the singularity reach the far away observer, the *weak cosmic censorship* is violated, (see [1, 6, 7] for reviews of the CCC). For an asymptotic observer the violation of strong censorship is of no harm, as long as the weak form is preserved. Both are equally serious violations of a natural law which forbid the visibility of singularity. But the formation of the singularity and its behavior, it seems, should be unaffected by the activity in the asymptotic region, and should depend only on the matter in a sufficiently local region or after crossing the event horizon.

The cosmic censorship conjecture has remained unproved despite many serious efforts. Part of the difficulty lies in not having a unique rigorous statement one may try to prove. Consequently, examples that appear to violate the CCC are very important and they are useful tools to study this important issue. There is one seemingly naked singularity obvious to everyone, the initial singularity of the *Big Bang*. This singularity is not really a counter example to the conjecture because it did not form from regular initial conditions and could not have done so according to the singularity theorems of general relativity. Studying gravitational collapse using Einstein equations is a formidable task, and evidence for formation of black hole is as limited as for a naked singularity.

2 Self-similarity

Colloquially self-similarity means part is similar to the whole. Profoundly, a self-similar space-time is characterized by the existence of a homothetic Killing vector field (Cahil and Taub, 1971). Such self-similarity is called the *first kind* (or homothety or continuous self-similarity). Cahill and Taub has showed that $p = k\rho$ is the only barotropic equation of state that is compatible with self-similarity of the first kind [8].
In general relativity, there exists a natural generalization of homothety called kinematic self-similarity, which is defined by the existence of a kinematic selfsimilar vector field [9] (see also the earlier related works by Tomita [10]). Kinematic self-similarity is characterized by an index and classified into three kinds: the *second* (or discrete self-similarity), *zeroth* and *infinite* kinds.

3 Self-similarity of First kind

The self-similarity of first kind admits a homothetic Killing vector field, that is, a vector field ξ such that, $L_\xi g_{\mu\nu} = 2g_{\mu\nu}$, where the notation $L_\xi g_{\mu\nu}$ indicates the Lie derivative of the metric, taken in the direction of the vector field ξ (to note that, the Lie derivative compares $g_{\mu\nu}$ at two different points along the integral curves of , and subtracts to construct a derivative. For the metric, the Lie derivative reduces to $L_\xi g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu}$). The choice of non-zero constant 2, on the right hand side above is arbitrary, and can be fixed by rescaling ξ.

A spherically symmetric space-time is self-similar if it admits a radial area coordinate r and an orthogonal time coordinate t such that for the metric components g_{tt} and g_{rr} , we have

$$g_{tt}(ct, cr) = cg_{tt}(t, r) \;;\; g_{rr}(ct, cr) = cg_{rr}(t, r) \text{ for all } c > 0.$$

Thus, along the integral curves of the Killing vector field, all points are similar [1].

The general form of a spherically symmetric metric in comoving coordinates can be written as

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\psi(t,r)}dr^2 + R^2(t, r)d\Omega^2, \tag{5}$$

where $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$ is the usual metric on a two-sphere. The imposition of self-similarity on this space-time results in considerable simplification. In particular, we have ν, ψ and S are functions of z only and $z = t/r$. More generally, self-similarity can be thought of as a symmetry which causes physical quantities to depend only on dimensionless parameters (which are typically ratios of time and space variables such as z).

The stress-energy tensor for a perfect fluid is given by

$$T_{ab} = [\rho(t,r) + P(t,r)]u_a u_b + P(t,r) g_{ab} \tag{6}$$

where four velocity of the fluid flow is $u^a = (e^{-\nu}, 0, 0, 0)$ and the pressure and energy density in the comoving coordinates are given by $P = p(z)/(8\pi r^2)$, $\rho = \eta(z)/(8\pi r^2)$. The self-similarity implies the existence of constants of motion along $dz = 0$, which in turn allows the reduction of the Einstein field equations $G_{ij} = 8\pi T_{ij}$ to a set of ordinary differential equations [11].

We assume that the collapsing fluid obeys an adiabatic equation of state

$$p(z) = k\ \eta(z) \tag{7}$$

where $0 \leq k \leq 1$ is a constant. Cahill and Taub has showed that $p = k\rho$ is the only barotropic equation of state that is compatible with self-similarity of the first kind [8].

From the Bianchi identity

$$T^{ab}_{;b} = 0 \tag{8}$$

it follows that

$$\dot{p} + \frac{2p}{z} = -(\eta + p)\dot{\nu} \tag{9}$$

$$\dot{\eta} = -(\eta + p)\left[\dot{\psi} + \frac{2\dot{S}}{S}\right] \tag{10}$$

Integration of Eqs. (9) and (10), respectively, gives

$$e^{2\nu} = \gamma(\eta z^2)^{-2k/(1+k)} \tag{11}$$

$$e^{2\psi} = \alpha(\eta)^{-2/(1+k)} S^{-4} \tag{12}$$

where α and γ are integration constants. Elimination of $\ddot{S}$ from set of ordinary differential equations associated with G_{tt} and G_{rr} leads to

$$\left(\frac{\dot{S}}{S}\right)^2 V + \frac{\dot{S}}{S}\left(\dot{V} + 2ze^{2\nu}\right) + e^{2\psi+2\nu}\left(-\eta - e^{-2\psi} + \frac{1}{S^2}\right) = 0 \tag{13}$$

and

$$\dot{V} = ze^{2\nu}\left[(\eta + p)e^{2\psi} - 2\right] = ze^{2\nu}(H - 2) \tag{14}$$

where the quantities V and H are defined as

$$V(z) = e^{2\psi} - z^2 e^{2\nu}, \qquad H = (\eta + p)e^{2\psi} \tag{15}$$

One can also write H as

$$H = 8\pi r^2 e^{2\psi}\left(T_1^1 - T_0^0\right) \tag{16}$$

The matter satisfy weak energy condition if and only if

$$T_{ab}\ v^a v^b \geq 0 \tag{17}$$

for all nonspacelike vector v^a. Thus for matter satisfying weak energy condition, it follows that $H(z) \geq 0$ for all z.

A self-similar space-time is characterized by the existence of a homothetic Killing vector:

$$\xi^a = r\frac{\partial}{\partial r} + t\frac{\partial}{\partial t} \tag{18}$$

which is given by the Lie derivative

$$\mathcal{L}_\xi g_{ab} = \xi_{a;b} + \xi_{b;a} = 2g_{ab} \tag{19}$$

where $\mathcal{L}$ denotes the Lie derivative. Let $K^a = dx^a/dk$ be the tangent vector to the null geodesics, where k is an affine parameter. Then

$$g_{ab}K^a K^b = 0 \tag{20}$$

It follows that along null geodesics, we have

$$\xi^a K_a = C \tag{21}$$

where C is a constant. From the above algebraic equation and the null condition (20), we get

$$re^{2\psi}K^r - te^{2\nu}K^t = C \tag{22}$$

$$-e^{2\nu}(K^t)^2 + e^{2\psi}(K^r)^2 = 0 \tag{23}$$

Solving these equations, we have exact expressions for K^t and K^r:

$$K^t = \frac{C\left[z \pm e^{2\psi}Q\right]}{r\left[e^{2\psi} - e^{2\nu}z^2\right]} \tag{24}$$

$$K^r = \frac{C\left[1 \pm z e^{2\nu} Q\right]}{r\left[e^{2\psi} - e^{2\nu} z^2\right]} \tag{25}$$

where $Q = \sqrt{e^{-2\psi-2\nu}} > 0$.

The nature of singularity is studied by employing the technique developed by Dwivedi and Joshi [12]. Radial null geodesics (θ and $\phi = const.$), by virtue of Eqs. (24) and (25), satisfy

$$\frac{dt}{dr} = \frac{z \pm e^{2\psi} Q}{1 \pm z e^{2\nu} Q} \tag{26}$$

At this point, we note that a curvature singularity forms at the origin ($t = 0,\ r = 0$), where the physical quantities like density diverges. This divergence of the density in this singularity results also in a divergence of curvature scalars there. The nature of the singularity (a naked singularity or a black hole) can be characterized by the existence of radial null geodesics emerging from the singularity. The singularity is at least locally naked if there exist such geodesics, and if no such geodesics exist it is a black hole. If the singularity is naked, then there exists a real and positive value of z_0 as a solution to the algebraic equation [7]

$$z_0 = \lim_{t\to 0\, r\to 0} z = \lim_{t\to 0\, r\to 0} \frac{t}{r} = \lim_{t\to 0\, r\to 0} \frac{dt}{dr} \tag{27}$$

Using (26) and L'Hôpital's rule we can derive the following equation

$$V(z_0)Q(z_0) = 0 \tag{28}$$

Since $Q > 0$, this implies that

$$V(z_0) = 0 \tag{29}$$

This algebraic equation governs the behavior of the tangent near the singular points. The central shell focusing is at least locally naked if Eq. (29) admits one or more positive roots. The values of the roots give the tangents of the escaping geodesics near the singularity. The smallest value of z_0, say z_0^s, corresponds to the earliest ray escaping from the singularity which, is called the Cauchy horizon of the space-time and there is no solution in the region $z < z_0^s$. Thus in the absence of a positive root to Eq. (29), the central singularity is not naked because there is no outgoing future directed null geodesics emanating from the singularity.

Further studying self-similar field equations, Joshi and Dwivedi found that the condition $V(z) = 0$ has real positive roots in terms of physical parameters of energy density η_0 and η_1 [1].

Also they have discussed global visibility of the naked singularity. A naked singularity can be considered as a physically significant singularity, if it can escape from singularity to far away observers for a finite period of time. They have shown that infinity of integral curves would escape the singularity provided the weak energy condition is fulfilled.

Further, we determine the curvature strength of the naked singularity, which is an important aspect of a singularity [13]. A singularity is gravitationally strong or simply strong if volume elements get crushed to zero dimensions at the singularity, and weak otherwise. It is widely believed that a space-time does not admit an extension through a singularity if it is a strong curvature singularity in the sense of Tipler [14]. A necessary and sufficient condition criterion for a singularity to be strong has been given by Clarke and Królak [15] that for at least one non-spacelike geodesic with affine parameter c, in the limiting approach to the singularity, we must have

$$\lim_{c\to 0} c^2\psi = \lim_{c\to 0} c^2 R_{ab}K^a K^b > 0 \tag{30}$$

where R_{ab} is the Ricci tensor. Our purpose here is to investigate the above condition along future directed radial null geodesics that emanate from the naked singularity. Eq. (30) can be expressed as

$$\lim_{c\to 0} c^2\psi = \frac{4H_0}{(H_0+2)^2} > 0 \tag{31}$$

where $H_0 = H(z_0)$ at the singularity $t = 0, r = 0, c = 0$ and $z = z_0$. Thus along radial null geodesics strong curvature condition is satisfied if $H_0 > 0$, which is also a necessary condition for the energy condition. Thus it follows that singularities are gravitationally strong if the weak energy condition is satisfied. Thus study of gravitational collapse of self-similar model of first kind reveals occurrence of globally NS as a counter example to CCC.

Ghosh, Sarwe and Saraykar studied the self-similar model of first kind and cosmic censorship, and extended 4-D results to 5-dimensional space-time [16].

4 Self-similarity of second, zeroth and infinite kind

In this section, we study the self-similarity models of various kinds in spherically symmetric space-time given by metric (5). We consider the matter field to be perfect fluid described by

$$T_{ij} = p(t,r)g_{ij} + [\rho(t,r) + p(t,r)]u_i u_j. \tag{32}$$

In terms of comoving coordinates, Einstein field equations and equations of motion of perfect fluid with units $G = 1, c = 1$ are written as [17]

$$\begin{aligned}
8\pi\rho &= \frac{1}{R^2} + e^{-2\nu}\left[\left(\frac{R_t}{R}\right)^2 + 2\psi_t \frac{R_t}{R}\right] \\
&\quad -e^{-2\psi}\left[2\frac{R_{rr}}{R} - 2\psi_r \frac{R_r}{R} + \left(\frac{R_r}{R}\right)^2\right] && \text{(33a)}\\
8\pi p &= -\frac{1}{R^2} + e^{-2\psi}\left[\left(\frac{R_r}{R}\right)^2 + 2\nu_r \frac{R_r}{R}\right] \\
&\quad -e^{-2\nu}\left[2\frac{R_{tt}}{R} - 2\nu_t \frac{R_t}{R} + \left(\frac{R_t}{R}\right)^2\right] && \text{(33b)}\\
8\pi p &= e^{-2\psi}\left[\nu_{rr} + \nu_r^2 - \nu_r\psi_r + \frac{R_{rr}}{R} + \frac{R_r\nu_r}{R} - \frac{R_r\psi_r}{R}\right] \\
&\quad -e^{-2\nu}\left[\psi_{tt} + \psi_t^2 - \nu_t\psi_t + \frac{R_{tt}}{R} + \frac{R_t\psi_t}{R} - \frac{R_t\nu_t}{R}\right] && \text{(33c)}\\
\nu_r &= -\frac{p_r}{\rho + p} && \text{(33d)}\\
\psi_t &= -\frac{\rho_t}{\rho + p} - 2\frac{R_t}{R} && \text{(33e)}\\
R_{tr} &= \nu_r R_t + \psi_t R_r && \text{(33f)}\\
m_r &= 4\pi\rho R_r R^2 && \text{(33g)}\\
m_t &= -4\pi p R_t R^2 && \text{(33h)}\\
2m &= R[1 + e^{-2\nu}R_t^2 - e^{-2\psi}R_r^2] && \text{(33i)}
\end{aligned}$$

where subscripts t and r denote derivative with respect to t and r, respectively and $m(t,r)$ is the Misner-Sharp mass.

Maeda et. al. assume two different kinds of polytropic equations of state (EoS): First is

$$p = \mathcal{K}\rho^{\gamma} \tag{34}$$

where $\mathcal{K}$ and γ are constants. The second EoS is

$$p = \mathcal{K}n^{\gamma} \text{ and } \rho = m_b\, n + \frac{p}{\gamma - 1} \tag{35}$$

where $\mathcal{K} \neq 0$, $\gamma \neq 0, 1$, the constant m_b and $n(t, r)$ correspond to the mean baryon mass and the baryon number density, respectively. The linear EoS is also considered, and called here as EoS III,

$$p = \mathcal{K}\rho \text{ where } -1 \leq \mathcal{K} \leq 1 \text{ and } \mathcal{K} \neq 0. \tag{36}$$

We note from EoS I and II that for $\gamma < 0$, the fluid suffers from thermodynamical instability. For $0 < \gamma < 1$, both EOS (I) and (II) are approximated by a dust fluid in high-density regime, since $p/\rho = \mathcal{K}\rho^{\gamma-1} \to 0$ for $\rho \to \infty$ for EoS I and for EoS II,

$$\frac{p}{\rho} = \frac{\mathcal{K}n^{\gamma-1}}{m_b + \frac{\mathcal{K}n^{\gamma-1}}{\gamma-1}} \to 0 \text{ for } n \to \infty$$

When $\gamma > 1$, $\mathcal{K} = \gamma - 1$ and since in high-density regime,

$$\rho = m_b + \frac{\mathcal{K}n^{\gamma-1}}{\gamma - 1} \to \frac{\mathcal{K}n^{\gamma-1}}{\gamma - 1} = \frac{p}{\gamma - 1} \text{ for } n \to \infty$$

, so then here EoS II is approximated by EoS III.
In the case of $\gamma > 2$ for EoS II and $\gamma > 1$ for EoS I, the dominant energy condition can be violated in high-density regime, which will be unphysical.

In a spherically symmetric spacetime, a vector field ζ is written in general as

$$\zeta^{\mu}\frac{\partial}{\partial x^{\mu}} = h_1(t, r)\frac{\partial}{\partial t} + h_2(t, r)\frac{\partial}{\partial r} \tag{37}$$

where h_1 and h_2 are arbitrary functions. Note that $h_2 = 0$ when ζ is parallel to the fluid flow, while $h_1 = 0$, when ζ is orthogonal to the fluid flow. When ζ is tilted, in this case both h_1 and h_2 are non-zero.

A kinematic self-similarity vector ξ satisfies the condition

$$L_\xi h_{\mu\nu} \quad = 2\delta h_{\mu\nu}, \tag{38a}$$

$$L_\xi U_\mu \quad = \alpha U_\mu, \tag{38b}$$

where $h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu$ is the projection tensor, α and δ are constants [9, 19].

The similarity transformation is characterized by the scale-independent ratio, α/δ, which is referred to as the similarity index. In the case of $\delta \neq 0$, if we set $\delta = 1$, the kinematic self-similar vector ξ^μ can be written as

$$\xi^\mu \frac{\partial}{\partial x^\mu} = (\alpha t + \beta)\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} \tag{39}$$

and in this case ξ^μ is said to be tilted.

The case of $\alpha = 1$, $\beta = 0$ corresponds to the self-similarity of the first kind (this is discussed in section 3), it follows that ξ^μ is a homothetic vector and the *self-similar variable* $\xi = r/t$.

The case of $\alpha = 0$, $\beta = 1$ corresponds to self-similarity of the *zeroth kind*, here the self-similar variable is given by

$$\xi = re^{-t} \tag{40}$$

In the case of $\alpha \neq 0, 1$, which is corresponding to self-similarity of the *second kind* ($\beta = 0$), the self-similar variable is given by

$$\xi = \frac{r}{(\alpha t)^{1/\alpha}}. \tag{41}$$

The special case of $\delta = 0$ and $\alpha \neq 0$, the self-similarity is referred to as the *infinite kind* ($\alpha = 1$ is possible). The kinematic self-similar vector ξ^μ can be written as

$$\xi^\mu \frac{\partial}{\partial x^\mu} = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} \tag{42}$$

and the self-similar variable $\xi = r/t$. in this case metric functions are written as

$$R = S(\xi),\ e^\nu = e^{\nu(\xi)},\ \ e^\psi = \frac{e^{\psi(\xi)}}{r} \tag{43}$$

In the case of $\delta \neq 0$, self-similarity implies that the metric functions can be written as

$$R = rS(\xi),\ \nu = \nu(\xi),\ \psi = \psi(\xi). \tag{44}$$

In the following subsections, we study tilted case of self-similarity.

4.1 Self-similarity of the second kind

In this case, Einstein field equations imply that the quantities m, ρ and p must be of the form

$$2m = r\left[M_1(\xi) + \frac{r^2}{t^2}M_2(\xi)\right], \tag{45a}$$

$$8\pi\rho = \frac{1}{r^2}\left[\rho_1(\xi) + \frac{r^2}{t^2}\rho_2(\xi)\right], \tag{45b}$$

$$8\pi p = \frac{1}{r^2}\left[P_1(\xi) + \frac{r^2}{t^2}P_2(\xi)\right] \tag{45c}$$

where $\xi = r/(\alpha t)^{1/\alpha}$. A set of ordinary differential equations is obtained when one demands that the Einstein equations and the equations of motion for the matter field are satisfied for the $O[(r/t)^0]$ and $O[(r/t)^2]$ terms separately. On solving the equations for a perfect fluid (33a - 33i), the solution is Schwarzschild spacetime in the Lemaitres choice of coordinates, written as [17]

$$ds^2 = -dt^2 + r_g^{2/3}\left[\frac{(9/4)dr^2}{[\frac{3}{2}(1-t/r^{3/2})]^{2/3}} + r^2\left[\frac{3}{2}(1-t/r^{3/2})\right]^{4/3}d\Omega^2\right] \tag{46}$$

where r_g is a constant and the Schwarzschild radius corresponds to $r_g = (3/2)(\rho - t)$ [18].

4.2 Self-similarity of zeroth kind

In this case, Einstein field equations imply that the physical quantities m, ρ and p should be of the form

$$2m = r\left[M_1(\xi) + r^2 M_2(\xi)\right], \tag{47a}$$

$$8\pi\rho = \frac{1}{r^2}\left[\rho_1(\xi) + r^2\rho_2(\xi)\right], \tag{47b}$$

$$8\pi p = \frac{1}{r^2}\left[P_1(\xi) + r^2 P_2(\xi)\right] \tag{47c}$$

where $\xi = re^{-t}$. On solving the equations for a perfect fluid (33a - 33i), (for details refer [17]) with EoS I and II, one obtains a vacuum space-time while for EoS III, space-time is the de-Sitter with metric

$$
\begin{aligned}
ds^2 &= -dt'^2 + e^{2\sqrt{-p_o/3t'}}(dr'^2 + r'^2 d\Omega^2), && \text{(48a)}\\
2m &= -\frac{p_o}{3} r'^3 e^{3\sqrt{-p_o/3t'}}, && \text{(48b)}\\
8\pi p &= -8\pi\rho = 8\pi p_o && \text{(48c)}
\end{aligned}
$$

where $p_o = -(3/c_o^2)(S'/S)^2$, $t' = c_o t$ and $r' = s_o r^{1-\sqrt{-c_o^2 p_o/3}}$, and c_o and s_o are constants.

4.3 Self-similarity of the infinite kind

In the case of self-similarity of infinite kind, field equations impose the conditions on the quantities m, ρ and p to have the form

$$
\begin{aligned}
2m &= \frac{M_1(\xi)}{t^2} + M_2(\xi), && \text{(49a)}\\
8\pi\rho &= \frac{\rho_1(\xi)}{t^2} + \rho_2(\xi), && \text{(49b)}\\
8\pi p &= \frac{P_1(\xi)}{t^2} + P_2(\xi) && \text{(49c)}
\end{aligned}
$$

where $\xi = r/t$. Following the usual procedure of solving the equations for a perfect fluid (33a - 33i) with using EoS III, the solution is

$$
\begin{aligned}
ds^2 &= -\frac{s_o^2}{c_1^2} t'^{2s_o/c_1 - 2}[A sin(ln\ r' - ln\ t') + B cos(ln\ r' - ln\ t')]^2 dt'^2 \\
&\quad + \frac{s_o^2}{r'^2}(dr'^2 + r'^2 d\Omega^2), && \text{(50a)}\\
2m &= -s_o, && \text{(50b)}\\
8\pi p &= -8\pi\rho = -\frac{1}{s_o^2} && \text{(50c)}
\end{aligned}
$$

where $t' = t^{c_1/s_o}$ and $r' = r^{c_1/s_o}$, c_1 and s_o are constants. This solution though not identical but closely related to the solution obtained by Nariai [20].

5 Self-similarity vector ξ^μ parallel to U^μ

5.1 Self-similarity of second kind

We can choose ξ^μ to be

$$\xi^\mu \frac{\partial}{\partial x^\mu} = t\frac{\partial}{\partial t} \tag{51}$$

and then the metric can be expressed as

$$ds^2 = -t^{2(\alpha-1)}e^{2\nu(r)}dt^2 + t^2dr^2 + S(r)^2t^2d\Omega^2. \tag{52}$$

The Einstein equations imply that the quantities m, ρ and p must be of the form

$$2m = tM_1(r) + t^{3-2\alpha}M_2(r), \tag{53a}$$
$$8\pi\rho = t^{-2}\rho_1(r) + t^{-2\alpha}\rho_2(r), \tag{53b}$$
$$8\pi p = t^{-2}P_1(r) + t^{-2\alpha}P_2(r) \tag{53c}$$

where $\xi = r$. The solution of set of field equations together with EoS I and II gives the metric

$$ds^2 = -dt'^2 + (-p_o)^{-1}\left(\frac{3}{2c_o}\right)^{4/3} t'^{4/3}(dr'^2 + sin^2r'd\Omega^2), \tag{54a}$$
$$2m = \left[-\left(\frac{3}{2}\right)^{2/3} c_o^{-2/3}(-p_o)^{-1/2}t'^{2/3} + c_o^{-2}(-p_o)^{-3/2}\right] sin^3r', \tag{54b}$$
$$8\pi p = \left(\frac{3}{2c_o}\right)^{-4/3} p_o t'^{-4/3}, \tag{54c}$$
$$8\pi\rho = -3p_o\left(\frac{3}{2c_o}\right)^{-4/3} t'^{-4/3} + \frac{4}{3}t'^{-2}. \tag{54d}$$

where transformations used are $t' = (2c_o/3)t^{3/2}$ and $r' = \sqrt{-p_o r}$. This solution is found to be the closed FRW solution with dust and comoving fluids with $p = \rho/3$.

5.2 Self-similarity of zeroth kind

As in the above section of self-similarity of second kind of parallel case, by keeping $\alpha = 0$ in equations of subsection (4.1), analysis with EoS III give rise to the de-Sitter solution. With the coordinate transformation $t = exp(\sqrt{-p_o/3t'})$, the solution is written as

$$ds^2 = -dt'^2 + e^{2\sqrt{-p_o/3t'}}(dr^2 + r^2 d\Omega^2), \tag{55a}$$

$$2m = \frac{-p_o}{3} r^3 e^{3\sqrt{-p_o/3t'}}, \tag{55b}$$

$$8\pi p = -8\pi\rho = 8\pi p_o. \tag{55c}$$

5.3 Self-similarity of infinite kind

The self-similarity vector ξ^μ can be chosen as

$$\xi^\mu \frac{\partial}{\partial x^\mu} = t\frac{\partial}{\partial t}. \tag{56}$$

Field equations impose conditions on the quantities m, ρ and p to have the form

$$2m = M(r), \tag{57a}$$

$$8\pi\rho = \rho(r), \tag{57b}$$

$$8\pi p = P(r) \tag{57c}$$

where $\xi = r$. Following the usual procedure of solving the equations for a perfect fluid equations (33a - 33i), give the Tolman-Oppenheimer-Volkoff equation. It implies that any spherically symmetric static spacetime is a self-similar solution of the infinite kind in which kinematic self-similar vector is parallel to the fluid flow. In a vacuum case, the Schwarzschild solution can be obtained [17].

6 Self-similarity vector ξ^μ orthogonal to U^μ

6.1 Self-similarity of the second kind

The self-similarity vector ξ^μ is written as

$$\xi^\mu \frac{\partial}{\partial x^\mu} = r\frac{\partial}{\partial r}. \tag{58}$$

The Einstein equations impose conditions that the quantities m, ρ and p must be of the form

$$
\begin{aligned}
2m &= rM_1(t) + r^{3-2\alpha}M_2(t), && (59a)\\
8\pi\rho &= r^{-2}\rho_1(t) + r^{-2\alpha}\rho_2(t), && (59b)\\
8\pi p &= r^{-2}P_1(t) + r^{-2\alpha}P_2(t) && (59c)
\end{aligned}
$$

It is noted that the solution is always singular at $r = 0$, which is correspond to the physical center. Analysis shows that there are no solutions with respect to EoS I and II. With the use of EoS III, the solution can be written as follows:

$$
\begin{aligned}
&ds^2 = -r^{2\alpha}dt^2 + \frac{s_o^2}{1 - w_o s_o^2} + s_o^2 r^2 d\Omega^2, && (60a)\\
&M_1 = w_o s_o^3, \quad M_2 = 0, && (60b)\\
&P_1 = \frac{\alpha}{2-\alpha}\rho_1 = \frac{\alpha}{2-\alpha} w_o, && (60c)\\
&P_2 = \rho_2 = 0. && (60d)
\end{aligned}
$$

where w_o and s_o are constants and $(1 + 2\alpha - \alpha^2)w_o s_o^2 = \alpha(2 - \alpha)$.

6.2 Self-similarity of the zeroth kind

In above sub-section, we can put $\alpha = 0$, to understand and analyze the self-similarity of zeroth kind. In a vacuum case, the Minkowski spacetime can be obtained since $M_1 = M_2 = 0$.

On using EoS I and II, we can have flat FRW metric

$$ds^2 = -dt^2 + S^2(dr^2 + r^2 d\Omega^2) \tag{61}$$

where the S is governed by equations

$$\rho_2 S^2 = 3S'^2, \quad -P_2 S^2 = 2S''S + S'^2. \tag{62}$$

In terms of EoS III, we have equations

$$P_1 = \mathcal{K}\rho_1, \quad P_2 = \mathcal{K}\rho_2. \tag{63}$$

Herein, there are two solutions namely flat FRW space-time for $\mathcal{K} \neq 1$ and $S = s_o t^{2/3(1+\mathcal{K})}$,

$$ds^2 = -dt^2 + t^{2/3(1+\mathcal{K})}(dr'^2 + r'^2 d\Omega^2), \tag{64a}$$

$$2m = \frac{4}{9(1+\mathcal{K})^2} t^{-2\mathcal{K}/(1+\mathcal{K})} r'^3, \tag{64b}$$

$$8\pi p = 8\pi\mathcal{K}\rho = \frac{4}{3(1+\mathcal{K})^2} t^{-2}. \tag{64c}$$

and the other solution is the de-Sitter solution when $\mathcal{K} = 1$ and $S = s_o e^{\sqrt{(-p_o/3t}}$:

$$ds^2 = -dt^2 + e^{2\sqrt{(-p_o/3t}}(dr'^2 + r'^2 d\Omega^2), \tag{65a}$$

$$2m = \frac{-p_o}{3} e^{3\sqrt{(-p_o/3t}} r'^3, \tag{65b}$$

$$8\pi p = -8\pi\rho = p_o. \tag{65c}$$

6.3 Self-similarity of the infinite kind

The self-similarity vector ξ^μ is expressed as

$$\xi^\mu \frac{\partial}{\partial x^\mu} = r\frac{\partial}{\partial r}. \tag{66}$$

The Einstein equations impose conditions that the quantities m, ρ and p must be of the form

$$2m = e^{-2r} M_1(t) + M_2(t), \tag{67a}$$

$$8\pi\rho = e^{-2r}\rho_1(t) + \rho_2(t), \tag{67b}$$

$$8\pi p = e^{-2r} P_1(t) + P_2(t). \tag{67c}$$

The resulting equations for a perfect fluid are written as:

$$S = M_2 = s_o, \tag{68a}$$

$$M_1 = 0, \tag{68b}$$

$$P_1 = \rho_1 = 0, \tag{68c}$$

$$P_2 = -\rho_2 = -w_o = -\frac{1}{s_o^2} \tag{68d}$$

where s_o and w_o are constants. Further analysis give the metric function as $e^{2\psi}S^{-2} = -1 < 0$, and therefore it can be concluded that there are no solutions in this case, independent of the form of the equation of state.

7 Conclusions

The study of gravitational collapse of self-similar model of first kind has been carried out with the purpose of deliberating on the aspects of spherically symmetric self-similar space-time, its structure and formation of naked singularity and its local/ global nature. The study reveals occurrence of globally NS as a counter example to CCC in the gravitational of collapse of the perfect fluid cloud.

The classification of the kinematic self-similar perfect-fluid solutions with either equation of state (I), (II) or (III) has been studied. In most cases, the governing equations can be integrated to give exact solutions, although there are a few exceptions. The analytic form of general solutions has not been obtained in the infinite-kind case with equation of state (III) for $K = 1$ in which a kinematic self-similar vector is tilted. It should also be noted that, independent of the form of the equation of state, kinematic self-similarity of the infinite kind in the orthogonal case is incompatible with a spherically symmetric spacetime.

In the cases of equation of state (I) and (II), (i.e., the polytropic equation of state), the FRW solution is one of the compatible solutions. The closed FRW solution with dust and $p = \rho/3$ comoving fluids is the second-kind solution in the parallel case for equation of state (II) with $\gamma = 2/3$, while the flat FRW solution is the zeroth-kind solution in the orthogonal case for both equation of state (I) and (II), in which the scale factor is not a power-law function of t in general.

These solutions further need to be studied to determine the strength of the singularity, spacetime structure near the singularity and the formation of BH and NS by analyzing the formation of apparent horizon.

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